ON CONVERGENCE FACTORS IN TRIPLE SERIES AND THE TRIPLE FOURIER'S SERIES.

By Bess M. Eversull.

Although the triple Fourier's series has been used somewhat extensively, no proof of the validity of the development of a function of three variables in such a series has been given. It is the purpose of this paper to establish certain facts in connection with the summability of such a series, with a view to applying the theory of convergence factors to some problems in mathematical physics. As an illustration of such applications it will be shown that the formal series arising from the discussion of a certain problem in the flow of heat furnishes an actual solution to the problem. A method of dealing with problems of this type has been originated by Fejér* and applied by him to problems involving the ordinary Fourier's series; the same method has been applied to problems involving the double Fourier's series by Professor C. N. Moore† and will here be applied to the problem involving the triple Fourier's series which we wish to consider.

1. **Summability of triple series.** The type of summability with which this paper is concerned will first be defined. Consider the triple series

(1)
$$\sum_{l=1,\, m=1,\, n=1}^{\infty,\, \infty,\, \infty} a_{lmn}$$

and form

(2)
$$S_{lmn}^{(r)} = \sum_{i=1,\, j=1,\, k=1}^{l,\, m,\, n} \frac{\Gamma(r+l-i)}{\Gamma(r)\Gamma(l-i+1)} \cdot \frac{\Gamma(r+m-j)}{\Gamma(r)\Gamma(m-j+1)} \cdot \frac{\Gamma(r+n-k)}{\Gamma(r)\Gamma(n-k+1)}\, s_{ijk},$$

where

(3)
$$s_{ijk} = \sum_{p=1,\, q=1,\, s=1}^{i,\, j,\, k} a_{pqs}.$$

If the quotient

(4)
$$\frac{S_{lmn}^{(r)}}{A_{lmn}^{(r)}},$$

* Cf. his paper *Untersuchungen über Fouriersche Reihen*, Math. Annalen, vol. 58 (1903–1904), p. 51.

† See his paper *On convergence factors in double series and the double Fourier's series*, Trans. Amer. Math. Soc., vol. 14 (1913), pp. 73–104. The summability of the double Fourier's series has also been considered by W. H. Young (Proc. Lond. Math. Soc., ser. 2, vol. 11 (1912), p. 133), and by W. W. Küstermann (cf. his dissertation, *Über Fouriersche Doppelreihen und das Poissonsche Doppelintegral;* Munich, 1913).

where

$$(5)\qquad A^{(r)}_{lmn}=\frac{\Gamma(l+r)}{\Gamma(r+1)\Gamma(l)}\cdot\frac{\Gamma(m+r)}{\Gamma(r+1)\Gamma(m)}\cdot\frac{\Gamma(n+r)}{\Gamma(r+1)\Gamma(n)},$$

approaches a limit S as l, m and n become infinite, the triple series is said to be summable (Cr) and to have a value equal to this limit. This type of summability for a triple series is analogous to that which Cesàro considered for an ordinary series. The above definition is valid for any r, real or complex, except zero or a negative integer. We may include the case where $r=0$, if we assume that the right hand sides of equations (2) and (5) have the values they approach as r approaches zero. In this case, summability $(C0)$ is the same as convergence as defined by Pringsheim. For the applications we wish to make, we need only consider the case where r is zero or a positive integer.

Before going on with our investigation, it will be advantageous to develop some properties of $S^{(r)}_{lmn}$ and $A^{(r)}_{lmn}$. From the definitions (2) and (5), it is evident that

$$(6)\quad \sum_{l=1,m=1,n=1}^{\infty\,\infty\,\infty} S^{(r)}_{lmn}x^l y^m z^n \backsim (1-x)^{-r}(1-y)^{-r}(1-z)^{-r}\sum_{l=1,m=1,n=1}^{\infty\,\infty\,\infty} s_{lmn}x^l y^m z^n.$$

$$(7)\quad \sum_{l=1,m=1,n=1}^{\infty\,\infty\,\infty} A^{(r)}_{lmn}x^l y^m z^n \backsim xyz(1-x)^{-(r+1)}(1-y)^{-(r+1)}(1-z)^{-(r+1)},$$

the sign $\backsim$ indicating that, if the expressions on each side of it are expanded in ascending powers of x, y and z, the coefficients of the corresponding terms will be equal.

We may also derive the relation*

$$(8)\quad a_{lmn}=\sum_{i=0,j=0,k=0}^{r+1,r+1,r+1}(-1)^i(-1)^j(-1)^k\binom{r+1}{r+1-i}\binom{r+1}{r+1-j}\cdot\binom{r+1}{r+1-k}S^{(r)}_{l-i,m-j,n-k},$$

where for the sake of uniformity we set

$$(9)\qquad S^{(r)}_{pqs}=0\qquad (p,\ q \text{ or } s\leqq 0).$$

* The derivation of this expression may be generalized from that for the simple series, given in Bromwich, *Theory of infinite series*, p. 317.

From the definition of $A^{(r)}_{lmn}$, we have

$$\lim_{l,m,n\to\infty} \frac{A^{(r)}_{lmn}}{l^r\, m^r\, n^r} = \left\{\frac{1}{\Gamma(r+1)}\right\}^3,$$

and therefore

$$(10) \qquad A^{(r)}_{lmn} < K l^r\, m^r\, n^r \qquad (l, m, n = 1, 2, 3, \ldots),$$

where K is a positive constant.

In order to make our definition of summability of wide use, it must be consistent with the definition of convergence; that is, if the series is summable $(C0)$, or convergent according to Pringsheim's definition, it should also be summable (Cr) for any $r > 0$, and to the same value. We shall prove that our definition satisfies this requirement for all classes of convergent triple series for which *the condition of finitude** is satisfied, i. e. for which

$$(11) \qquad |s_{lmn}| = \left|\frac{S^{(0)}_{lmn}}{A^{(0)}_{lmn}}\right| < C \qquad (l, m, n = 1, 2, 3, \ldots),$$

C being a positive constant. It is possible that this restriction is more narrow than necessary, but this question will not be considered here as the theorems on convergence factors which we shall have to prove present the same restriction.

If a convergent triple series satisfies the condition (11), it also possesses the property that as l, m and n become infinite, the quotient $S^{(r)}_{lmn}/A^{(r)}_{lmn}$ converges to the same value as s_{lmn}, and will also remain less in absolute value than the positive constant C for all values of l, m and n. Hence, to insure consistency with the definition of convergence, we shall consider only triple series which satisfy the conditions that

$$(12) \qquad \lim_{l,m,n\to\infty} \frac{S^{(r)}_{lmn}}{A^{(r)}_{lmn}} \text{ exists,}$$

and

$$(13) \qquad \left|\frac{S^{(r)}_{lmn}}{A^{(r)}_{lmn}}\right| < C \qquad (l, m, n = 1, 2, 3, \ldots)$$

where C is a positive integer, and r a positive integer or zero.

* This term was introduced by Bromwich and Hardy in their paper, *Some extensions to multiple series of Abel's Theorem*, Proc. Lond. Math. Soc., ser. 2, vol. 2 (1904), p. 161.

Before proceeding to the discussion of the theorem of consistency, it would be well to note that the $(r+1)^{\text{th}}$ difference of the triple sequence f_{ijk} $(i, j, k = 1, 2, 3, \ldots)$ is entirely analogous to the $(r+1)^{\text{th}}$ difference of the simple sequence, and is defined by

$$(14) \quad \triangle^{r+1}_{\substack{r+1\\r+1}} f_{ijk} = \sum_{p=0,\,q=0,\,s=0}^{r+1,\,r+1,\,r+1} (-1)^p(-1)^q(-1)^s$$

$$\cdot \binom{r+1}{r+1-p}\binom{r+1}{r+1-q}\binom{r+1}{r+1-s} f_{i+p,\,j+q,\,k+s}.$$

We shall now proceed to the discussion of the consistency theorem, proving first two necessary lemmas.

LEMMA 1. *If the two triple sequences*

$$(15) \qquad a_{lmn},\ b_{lmn} \qquad (l, m, n = 1, 2, 3, \ldots)$$

satisfy the conditions

$$\text{(a)} \qquad \left|\frac{\triangle^{1}_{\substack{1\\1}} a_{lmn}}{\triangle^{1}_{\substack{1\\1}} b_{lmn}}\right|^{*} < C, \qquad (l, m, n = 0, 1, 2, 3, \ldots),$$

$$\text{(b)} \qquad \triangle^{1}_{\substack{1\\1}} b_{lmn} < 0, \qquad (l, m, n = 1, 2, 3, \ldots),$$

$$\text{(c)} \qquad \lim_{l, m, n \to \infty} b_{lmn} = \infty,$$

$$\text{(d)} \qquad \begin{array}{lll} \displaystyle\lim_{p\to\infty} \frac{b_{\lambda qs}}{b_{pqs}} = 0, & \displaystyle\lim_{q\to\infty} \frac{b_{p\mu s}}{b_{pqs}} = 0, & \displaystyle\lim_{s\to\infty} \frac{b_{pq\nu}}{b_{pqs}} = 0, \\ \displaystyle\lim_{p, q\to\infty} \frac{b_{\lambda\mu s}}{b_{pqs}} = 0, & \displaystyle\lim_{p, s\to\infty} \frac{b_{\lambda q\nu}}{b_{pqs}} = 0, & \displaystyle\lim_{q, s\to\infty} \frac{b_{p\mu\nu}}{b_{pqs}} = 0, \end{array}$$

$$(\lambda, \mu, \nu = 1, 2, 3, \ldots),$$

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* In forming $\triangle^{1}_{\substack{1\\1}} a_{lmn}$ and $\triangle^{1}_{\substack{1\\1}} b_{lmn}$, for the sake of uniformity we set

$$a_{lmn} = 0 = b_{lmn} \qquad (l, m \text{ or } n \leqq 0).$$

where the limits are approached uniformly, the first for all positive integral values of q and s, the second for all positive integral values of p and s, the third of p and q, the fourth of s, the fifth of q and the sixth of p, and

$$\text{(e)} \qquad \lim_{l,m,n\to\infty} \frac{\Delta_{1\,1}^{\,1\,1} a_{lmn}}{\Delta_{1\,1}^{\,1\,1} b_{lmn}} \text{ exists,}$$

then we shall have

$$(16) \qquad \left|\frac{a_{lmn}}{b_{lmn}}\right| < C, \qquad (l, m, n = 1, 2, 3, \ldots),$$

and

$$(17) \qquad \lim_{l,m,n\to\infty} \frac{a_{lmn}}{b_{lmn}}$$

will exist and be equal to the limit in (e).

We shall first establish the inequality (16). From conditions (a) and (b), we have

$$(18) \qquad \left|\Delta_{1\,1}^{\,1\,1} a_{lmn}\right| < -C\,\Delta_{1\,1}^{\,1\,1} b_{lmn},$$

and therefore

$$(19) \qquad \sum_{i=0,j=0,k=0}^{l-1,m-1,n-1} \left|\Delta_{1\,1}^{\,1\,1} a_{ijk}\right| < -C \sum_{i=0,j=0,k=0}^{l-1,m-1,n-1} \Delta_{1\,1}^{\,1\,1} b_{ijk}.$$

From this and condition (b),

$$(20) \qquad \frac{\displaystyle\sum_{i=0,j=0,k=0}^{l-1,m-1,n-1} \left|\Delta_{1\,1}^{\,1\,1} a_{ijk}\right|}{\displaystyle\sum_{i=0,j=0,k=0}^{l-1,m-1,n-1} \left|\Delta_{1\,1}^{\,1\,1} b_{ijk}\right|} < C.$$

In view of the fact that $a_{lmn} = 0$ when l, m or $n \leqq 0$, it is obvious that

$$\sum_{i=0,j=0,k=0}^{l-1,m-1,n-1} \Delta_{1\,1}^{\,1\,1} a_{ijk} = -a_{lmn}; \qquad \sum_{i=0,j=0,k=0}^{l-1,m-1,n-1} \Delta_{1\,1}^{\,1\,1} b_{ijk} = -b_{lmn},$$

and therefore, making use of (20), we have

$$\left|\frac{a_{lmn}}{b_{lmn}}\right| = \left|\frac{\sum\limits_{i=0,j=0,k=0}^{l-1,m-1,n-1}\overset{1}{\underset{1}{\triangle}}{}^{1}\,a_{ijk}}{\sum\limits_{i=0,j=0,k=0}^{l-1,m-1,n-1}\overset{1}{\underset{1}{\triangle}}{}^{1}\,b_{ijk}}\right| \leqq \frac{\sum\limits_{i=0,j=0,k=0}^{l-1,m-1,n-1}\left|\overset{1}{\underset{1}{\triangle}}{}^{1}\,a_{ijk}\right|}{\sum\limits_{i=0,j=0,k=0}^{l-1,m-1,n-1}\left|\overset{1}{\underset{1}{\triangle}}{}^{1}\,b_{ijk}\right|}. \tag{21}$$

We have thus proved that the relation (16) is true, and it remains only to be shown that the limit (17) exists and is equal to the limit in (e). Since the limit in (e) exists, we can find a λ, μ and ν corresponding to a positive ε as small as we please, such that

$$L-\varepsilon < \frac{\overset{1}{\underset{1}{\triangle}}{}^{1}\,a_{lmn}}{\overset{1}{\underset{1}{\triangle}}{}^{1}\,b_{lmn}} < L+\varepsilon, \qquad \begin{pmatrix} l \geqq \lambda \\ m \geqq \mu \\ n \geqq \nu \end{pmatrix}, \tag{22}$$

where L is the value of the limit in (e). Hence from condition (b) it follows that

$$(L-\varepsilon)\overset{1}{\underset{1}{\triangle}}{}^{1}\,b_{lmn} > \overset{1}{\underset{1}{\triangle}}{}^{1}\,a_{lmn} > (L+\varepsilon)\overset{1}{\underset{1}{\triangle}}{}^{1}\,b_{lmn} \qquad \begin{pmatrix} l \geqq \lambda \\ m \geqq \mu \\ n \geqq \nu \end{pmatrix}. \tag{23}$$

If we add all the inequalities (23) for all sets of values of l, m and n such that $\lambda \leqq l \leqq p$, $\mu \leqq m \leqq q$, $\nu \leqq n \leqq s$, we have

$$\begin{aligned} &(L-\varepsilon)\,(b_{\lambda\mu\nu}-b_{p\mu\nu}-b_{\lambda q\nu}-b_{\lambda\mu s}+b_{pq\nu}+b_{p\mu s}+b_{\lambda qs}-b_{pqs}) \\ &\qquad > a_{\lambda\mu\nu}-a_{p\mu\nu}-a_{\lambda q\nu}-a_{\lambda\mu s}+a_{pq\nu}+a_{p\mu s}+a_{\lambda qs}-a_{pqs} \\ &> (L+\varepsilon)\,(b_{\lambda\mu\nu}-b_{p\mu\nu}-b_{\lambda q\nu}-b_{\lambda\mu s}+b_{pq\nu}+b_{p\mu s}+b_{\lambda qs}-b_{pqs}). \end{aligned}$$

From condition (c) it is obvious that we can find a p, q and s so large that $b_{pqs}>0$, and hence we may divide the above inequality by b_{pqs}, which gives

$$\begin{aligned} &(L-\varepsilon)\left[1-\frac{b_{\lambda qs}}{b_{pqs}}-\frac{b_{p\mu s}}{b_{pqs}}-\frac{b_{pq\nu}}{b_{pqs}}+\frac{b_{\lambda\mu s}}{b_{pqs}}+\frac{b_{\lambda q\nu}}{b_{pqs}}+\frac{b_{p\mu\nu}}{b_{pqs}}+\frac{b_{\lambda\mu\nu}}{b_{pqs}}\right] \\ &< \frac{a_{pqs}}{b_{pqs}}-\frac{a_{\lambda qs}}{b_{pqs}}-\frac{a_{p\mu s}}{b_{pqs}}-\frac{a_{pq\nu}}{b_{pqs}}+\frac{a_{\lambda\mu s}}{b_{pqs}}+\frac{a_{\lambda q\nu}}{b_{pqs}}+\frac{a_{p\mu\nu}}{b_{pqs}}+\frac{a_{\lambda\mu\nu}}{b_{pqs}} \\ &< (L-\varepsilon)\left[1-\frac{b_{\lambda qs}}{b_{pqs}}-\frac{b_{p\mu s}}{b_{pqs}}-\frac{b_{pq\nu}}{b_{pqs}}+\frac{b_{\lambda\mu s}}{b_{pqs}}+\frac{b_{\lambda q\nu}}{b_{pqs}}+\frac{b_{p\mu\nu}}{b_{pqs}}+\frac{b_{\lambda\mu\nu}}{b_{pqs}}\right]. \end{aligned} \tag{24}$$

From conditions (c) and (d) it follows that, as p, q and s become infinite, the limits of the parentheses in (24) which involve only b's exist and are equal to unity. The last seven terms of the second member of the inequality (24) may be written in the form

$$-\frac{a_{\lambda qs}}{b_{\lambda qs}}\cdot\frac{b_{\lambda qs}}{b_{pqs}}-\frac{a_{p\mu s}}{b_{p\mu s}}\cdot\frac{b_{p\mu s}}{b_{pqs}}-\frac{a_{pq\nu}}{b_{pq\nu}}\cdot\frac{b_{pq\nu}}{b_{pqs}}+\frac{a_{\lambda\mu s}}{b_{\lambda\mu s}}\cdot\frac{b_{\lambda\mu s}}{b_{pqs}}$$

$$+\frac{a_{\lambda q\nu}}{b_{\lambda q\nu}}\cdot\frac{b_{\lambda q\nu}}{b_{pqs}}+\frac{a_{p\mu\nu}}{b_{p\mu\nu}}\cdot\frac{b_{p\mu\nu}}{b_{pqs}}-\frac{a_{\lambda\mu\nu}}{b_{\lambda\mu\nu}}\cdot\frac{b_{\lambda\mu\nu}}{b_{pqs}};$$

hence, by virtue of conditions (c) and (d) and the inequality (21), the limit as p, q and s become infinite, of these seven terms, exists and is equal to zero.

Hence, from (24), we have

$$L-\epsilon \leqq \underline{\lim}\frac{a_{pqs}}{b_{pqs}} \leqq \overline{\lim}\frac{a_{pqs}}{b_{pqs}} \leqq L+\epsilon, \tag{25}$$

and since ϵ is an arbitrarily small positive quantity, it follows from (25) that

$$\underline{\lim}\frac{a_{pqs}}{b_{pqs}} = \overline{\lim}\frac{a_{pqs}}{b_{pqs}} = L,$$

and consequently the limit (17) exists and is equal to L, the limit in (e), and therefore the lemma is proved.

LEMMA 2. *If the triple series* (1) *is summable* (Cr) *where* r *is zero or a positive integer, and if moreover*

$$\left|\frac{S_{lmn}^{(r)}}{A_{lmn}^{(r)}}\right| < C, \qquad (l, m, n = 1, 2, 3, \ldots), \tag{26}$$

where C *is a positive constant, the series will be summable* $(C\,\overline{r+1})$ *to the same value to which it is summable* (Cr), *and furthermore we shall have*

$$\left|\frac{S_{lmn}^{(r+1)}}{A_{lmn}^{(r+1)}}\right| < C, \qquad (l, m, n = 1, 2, 3, \ldots). \tag{27}$$

From relations (6) and (7) we have

$$\sum_{l=1,\, m=1,\, n=1}^{\infty,\,\infty,\,\infty} S_{lmn}^{(r+1)} x^l y^m z^n \backsim (1+x+x^2+\cdots)(1+y+y^2+\cdots)$$
$$\cdot(1+z+z^2+\cdots) \sum_{l=1,\, m=1,\, n=1}^{\infty,\,\infty,\,\infty} S_{lmn}^{(r)} x^l y^m z^n,$$

$$\sum_{l=1,\, m=1,\, n=1}^{\infty,\,\infty,\,\infty} A_{lmn}^{(r+1)} x^l y^m z^n \backsim (1+x+x^2+\cdots)(1+y+y^2+\cdots)$$
$$\cdot(1+z+z^2+\cdots) \sum_{l=1,\, m=1,\, n=1}^{\infty,\,\infty,\,\infty} A_{lmn}^{(r)} x^l y^m z^n.$$

Equating the coefficients of the corresponding terms in each of these expressions, we have

$$(28) \qquad S_{lmn}^{(r+1)} = \sum_{i=1,\, j=1,\, k=1}^{l,\, m,\, n} S_{ijk}^{(r)}; \qquad A_{lmn}^{(r+1)} = \sum_{i=1,\, j=1,\, k=1}^{l,\, m,\, n} A_{ijk}^{(r)},$$

and from these equations we obtain

$$(29) \qquad \triangle_{\substack{1\\1}}^{1} S_{lmn}^{(r+1)} = -S_{l+1,\, m+1,\, n+1}^{(r)}; \qquad \triangle_{\substack{1\\1}}^{1} A_{lmn}^{(r+1)} = -A_{l+1,\, m+1,\, n+1}^{(r)}.$$

We may now apply Lemma 1, taking $S_{lmn}^{(r+1)}$ and $A_{lmn}^{(r+1)}$ as the two triple sequences. It is necessary first to show that the conditions under which Lemma 1 holds true are satisfied. Condition (a) is satisfied; for, from (29),

$$\left| \frac{\triangle_{\substack{1\\1}}^{1} S_{lmn}^{(r+1)}}{\triangle_{\substack{1\\1}}^{1} A_{lmn}^{(r+1)}} \right| = \left| \frac{S_{l+1,\, m+1,\, n+1}^{(r)}}{A_{l+1,\, m+1,\, n+1}^{(r)}} \right|,$$

which from (26) is less than a positive constant $C\,(l,\ m,\ n = 1, 2, 3, \ldots)$. Condition (b) is satisfied by virtue of the second of equations (29) and the definition (5). That conditions (c) and (d) are fulfilled also follows from definition (5). That condition (e) is satisfied follows from (29) and the hypothesis of this lemma that the triple series is summable (Cr).

Hence it follows from Lemma 1 that $S_{lmn}^{(r+1)}/A_{lmn}^{(r+1)}$ satisfies the condition of finitude, and that the series is summable $(C\,\overline{r+1})$ to the same value to which it is summable (Cr).

We may now prove the consistency theorem itself.

THEOREM I. *If the series* (1) *is summable* (Cr), *where* r *is zero or a positive integer, and*

$$\left|\frac{S_{lmn}^{(r)}}{A_{lmn}^{(r)}}\right| < C, \qquad (l, m, n = 1, 2, 3, \ldots),$$

where C *is a positive constant, then the series will be summable* (Cr'), *where* r' *is any integer greater than* r, *to the same value to which it is summable* (Cr), *and furthermore we shall have*

$$\left|\frac{S_{lmn}^{(r')}}{A_{lmn}^{(r')}}\right| < C \qquad (l, m, n = 1, 2, 3, \ldots).$$

If $r' = r + 1$, the theorem reduces to Lemma 2. If $r' > r + 1$, the theorem may be proved by successive applications of Lemma 2.

2. **Convergence factors in triple series.** Before proceeding with the work of this section, it will be necessary to introduce and define a notation which we shall need. If we set

$$(30) \quad \triangle_{r+1,u}^{\substack{r+1,w \\ r+1,v}} = \sum_{p=0,\,q=0,\,s=0}^{r+1-u,\,r+1-v,\,r+1-w} (-1)^p\,(-1)^q\,(-1)^s$$

$$\cdot\binom{r+1}{r+1-p}\binom{r+1}{r+1-q}\binom{r+1}{r+1-s} f_{i+p,j+q,k+s},$$

we see that it is analogous to the notation (14), and also that the right hand side of (30) is equal to that of (14) with certain of its terms suppressed; the terms which are suppressed are indicated by the indices added to those of expression (14). We may further abbreviate our notation by setting

$$\triangle_{r+1,u}^{\substack{r+1,r+1 \\ r+1,v}} = \triangle_{r+1,u}^{\substack{0 \\ r+1,v}}, \text{ etc.}$$

We are now ready to prove the theorems on convergence factors, restricting ourselves in this paper to the treatment of convergence factors in triple series which are summable $(C1)$, as this is the only case necessary for the applications we wish to make.

LEMMA 3. *If the triple series* (1) *is summable* $(C1)$ *and, moreover, the condition* (13) *is satisfied for* $r = 1$, *then, for all positive values of* α, β *and* γ, *the series*

$$(31) \qquad \sum_{l=1,\,m=1,\,n=1}^{\infty\;\infty\;\infty} a_{lmn}\, f_{lmn}(\alpha, \beta, \gamma)$$

will converge and have the same value as the series

$$(32) \qquad \sum_{i=1, j=1, k=1}^{\infty\, \infty\, \infty} S_{ijk}^{(1)} \triangle_2^{\frac{2}{2}} f_{ijk}(\alpha, \beta, \gamma),$$

which will also be convergent, provided the convergence factors $f_{ijk}(\alpha, \beta, \gamma)$ *satisfy the conditions*

$$(a) \qquad \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} ijk \left| \triangle_2^{\frac{2}{2}} f_{ijk}(\alpha, \beta, \gamma) \right| < K, \qquad (\alpha, \beta, \gamma > 0),$$

$$(b) \qquad \lim_{l \to \infty} l \sum_{j=1, k=1}^{\infty, \infty} jk \, |f_{ljk}(\alpha, \beta, \gamma)| = 0, \qquad (\alpha, \beta, \gamma > 0)$$

and the two other conditions of the same type,

$$(c) \qquad \lim_{l \to \infty, m \to \infty} lm \sum_{k=1}^{\infty} K \, |f_{lmk}(\alpha, \beta, \gamma)| = 0, \qquad (\alpha, \beta, \gamma) > 0),$$

and the two other conditions of this type, and

$$(d) \qquad \lim_{l \to \infty, m \to \infty, n \to \infty} [l m n \, f_{lmn}(\alpha, \beta, \gamma)] = 0, \qquad (\alpha, \beta, \gamma > 0),$$

it being understood that the limiting processes indicated by conditions (a) — (d) *all have a meaning.*

Substituting in the expression

$$(33) \qquad \sum_{i=1, j=1, k=1}^{l, m, n} a_{ijk} f_{ijk}(\alpha, \beta, \gamma)$$

the value of a_{ijk} given by putting $r = 1$ in (8), and rearranging the terms, we have

$$(34) \qquad \sum_{i=1, j=1, k=1}^{l, m, n} a_{ijk} f_{ijk} = \sum_{i=1, j=1, k=1}^{l-2, m-2, n-2} S_{ijk}^{(1)} \triangle_2^{\frac{2}{2}} f_{ijk} + R_{ijk}^{(1)},$$

where $R_{ijk}^{(1)}$ consists of three terms of each of the types

$$\sum_{j=1,k=1}^{m-2,n-2} S_{l-1,j,k}^{(1)} \triangle_{2,1}^{\,2}{}_{\,2} f_{l-1,j,k}, \sum_{j=1,k=1}^{m-2,n-2} S_{ljk}^{(1)} \triangle_{0}^{\,2}{}_{\,2} f_{ljk}, \sum_{k=1}^{n-2} S_{l-1,m-1,k}^{(1)} \triangle_{2,1}^{\,2,1}{}_{\,2} f_{l-1,m-1,k},$$

$$\sum_{k=1}^{n-2} S_{l-1,m,k}^{(1)} \triangle_{2,1}^{\,0}{}_{\,2} f_{l-1,m,k}, \sum_{k=1}^{n-2} S_{l,m-1,k}^{(1)} \triangle_{0}^{\,2,1}{}_{\,2} f_{l,m-1,k},$$

$$\sum_{k=1}^{n-2} S_{lmk}^{(1)} \triangle_{0}^{\,0}{}_{\,2} f_{lmk}, \quad S_{l-1,m-1,n}^{(1)} \triangle_{2,1}^{\,2,1}{}_{\,0} f_{l-1,m-1,n}, \quad S_{l-1,m,n}^{(1)} \triangle_{2,1}^{\,0}{}_{\,0} f_{l-1,m,n-1},$$

and in addition the terms $S_{l-1,m-1,n-1}^{(1)} \triangle_{2,1}^{\,2,1}{}_{\,2,1} f_{l-1,m-1,n-1}$ and $S_{lmn}^{(1)} f_{lmn}$.

We shall now show that the right-hand side of (34) approaches a limit as l, m and n become infinite for all positive values of α, β and γ, and that the second term approaches the limit zero. Then it will follow that the left-hand side approaches a limit under the same conditions, and that these limits will be equal. Hence we may conclude that the series (31) converges to the same value as the series (32), the value of (32) being the limiting value of the first term in the expansion (34).

The first term of the right-hand side of (34) is the sum of the $(l-2)(m-2)(n-2)$ terms of the series (32), contained in a rectangular parallelopiped of dimensions $l-2$, $m-2$, $n-2$, taken from the upper, forward, left-hand corner of the series. From condition (13) where $r=1$, we have for the general term of (32)

$$(35) \qquad \left| S_{lmn}^{(1)} \triangle_{2}^{\,2}{}_{\,2} f_{lmn} \right| < Clmn \left| \triangle_{2}^{\,2}{}_{\,2} f_{lmn} \right| \qquad \left(\begin{matrix} l,\ m,\ n = 1, 2, 3 \ldots \\ \alpha, \beta, \gamma > 0 \end{matrix} \right).$$

From condition (a) it follows that the series whose general term is the right-hand side of (35) converges, and therefore the series (32) is absolutely convergent, and the first term of (34) approaches as a limit the value to which the series (32) converges as l, m and n become infinite.

By expanding each of the terms, except the last, of $R_{ijk}^{(1)}$, and applying conditions (13) for $r=1$, and the appropriate one of the conditions (b)–(d), we may show that these terms approach zero as l, m and n become infinite. That the last term of $R_{ijk}^{(1)}$ approaches zero as l, m and n become infinite may be shown by applying condition (13) for $r=1$, and the condition (d). Hence $R_{ijk}^{(1)}$ approaches the limit zero as l, m and n become infinite, and the lemma is proved.

We are now ready to prove the theorem:

THEOREM II. *If the triple series* (1) *satisfies the conditions of Lemma 3, and the convergence factors $f_{ijk}(\alpha, \beta, \gamma)$ satisfy the conditions* (a)—(d) *inclusive of that lemma, and the additional conditions that*

(e) $f_{ijk}(\alpha, \beta, \gamma)$ *is continuous in* α, β *and* γ $\qquad \left(\begin{matrix} i, j, k = 1, 2, 3, \ldots \\ \alpha, \beta, \gamma > 0 \end{matrix}\right)$,

(f) $\lim\limits_{\alpha, \beta, \gamma \to 0} [f_{ijk}(\alpha, \beta, \gamma)] = f_{ijk}(0, 0, 0) = 1 \qquad (i, j, k = 1, 2, 3, \ldots)$,

(g) $\lim\limits_{\alpha, \beta, \gamma \to 0} \sum\limits_{i=1}^{i=\infty} i \left| \triangle_2^{\frac{2}{2}} f_{ijk}(\alpha, \beta, \gamma) \right| = 0$ *for every j and k, and the two other conditions of the same type,*

(h) $\lim\limits_{\alpha, \beta, \gamma \to 0} \sum\limits_{i=1, j=1}^{i=\infty, j=\infty} ij \left| \triangle_2^{\frac{2}{2}} f_{ijk}(\alpha, \beta, \gamma) \right| = 0$ *for every k, and the two other conditions of this type, and*

(i) $\sum\limits_{i=1, j=1, k=1}^{\infty, \infty, \infty} ijk \left| \triangle_2^{\frac{2}{2}} f_{ijk} \right|$ *is uniformly convergent in* $(\alpha \geqq \alpha_0 > 0,\ \beta \geqq \beta_0 > 0,\ \gamma \geqq \gamma_0 > 0)$,

then the series (31) *will define a function of α, β and γ, $F(\alpha, \beta, \gamma)$ which is continuous for all positive values of α, β and γ, and for which*

$$\lim_{\alpha, \beta, \gamma \to +0} [F(\alpha, \beta, \gamma)] = S, \tag{36}$$

where S is the value of the series (1).

From Lemma 3, we know that the series (31) converges to the same value as the series (32) for all positive values of α, β and γ. Hence if we can show that the series (32) defines a function of α, β and γ, which is continuous for all positive values of α, β and γ, and for which equation (36) holds true, our theorem will have been proved.

Since by hypothesis, the series is summable $(C1)$ to S, we may write

$$\frac{S_{lmn}^{(1)}}{lmn} = S + \epsilon_{lmn} \qquad \left(\lim_{l, m, n \to \infty} \epsilon_{lmn} = 0\right). \tag{37}$$

Using (37), we can reduce the series (32) to the form

$$S \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} ijk \triangle_2^{\frac{2}{2}} f_{ijk}(\alpha, \beta, \gamma) + \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} ijk\, \epsilon_{ijk} \triangle_2^{\frac{2}{2}} f_{ijk}(\alpha, \beta, \gamma). \tag{38}$$

We can evaluate the series in the first term of (38) by applying Lemma 3 to the triple series for which

$$(39) \qquad a_{111} = 1, \quad a_{ijk} = 0 \qquad (i, j \text{ or } k > 1),$$

observing that this series is convergent and satisfies the restriction (11). Hence by Theorem 1 this series is summable ($C1$), and therefore we can say that for this series the equation

$$(40) \qquad \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} a_{ijk} f_{ijk}(\alpha, \beta, \gamma) = \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} S_{ijk}^{(1)} \triangle_2^{\overset{2}{2}} f_{ijk}(\alpha, \beta, \gamma),$$

which expresses the equality between the expressions (31) and (32), reduces to the form

$$(41) \qquad f_{111}(\alpha, \beta, \gamma) = \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} ijk \triangle_2^{\overset{2}{2}} f_{ijk}(\alpha, \beta, \gamma).$$

Substituting this value in the expression (38), that expression becomes

$$(42) \qquad S f_{111}(\alpha, \beta, \gamma) + \sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} ijk\, \varepsilon_{ijk} \triangle_2^{\overset{2}{2}} f_{ijk}(\alpha, \beta, \gamma).$$

The first term is a continuous function of α, β and γ for all positive values of α, β and γ, by virtue of condition (e), and by condition (f) approaches the value S as α, β and γ approach zero from the positive direction. Hence if we can show that the second term of the expression (42), which is equal to the expression (32), is a continuous function of α, β and γ for all positive values of those variables, and approaches zero as α, β and γ approach zero, we shall have proved that the expression (42) is continuous for all positive values of α, β and γ, and approaches S as a limit as α, β and γ approach zero through positive values, and the theorem will have been proved.

By virtue of condition (i) and equations (11) and (37), we infer that the series in the second term of (42) is uniformly convergent in the region

$$(43) \qquad (\alpha \geqq \alpha_0 > 0, \ \beta \geqq \beta_0 > 0, \ \gamma \geqq \gamma_0 > 0).$$

From condition (e) its terms are continuous there, and therefore the series will be continuous in the region (43), and consequently, since α_0, β_0 and γ_0 are arbitrary positive quantities, will be continuous for all positive values

of α, β and γ. That it approaches zero as α, β and γ approach zero follows from the fact that ϵ_{ijk} remains finite for all values of i, j and k, and approaches zero as i, j and k become infinite, together with conditions (f), (g) and (h). Therefore, as pointed out above, the theorem has been proved.

The above theorem has been proved only for the case where the series (1) consists of constant terms. By extending the proofs of Lemma 3 and Theorem II, they may be made to apply to the case where the series (1) consists of terms which are functions of three variables. The analogous statement for this case is made in the following corollary:

COROLLARY. *If the triple series* (1), *whose terms are functions of x, y and z, is uniformly summable* ($C1$) *to $f(x, y, z)$ throughout the region R, and*

$$\left|\frac{S^{(1)}_{lmn}(x, y, z)}{lmn}\right| < C, \qquad \left(\underset{R}{l, m, n = 1, 2, 3, \ldots}\right),$$

where C is a positive constant, and the convergence factors $f_{ijk}(\alpha, \beta, \gamma)$ satisfy the conditions of Theorem II, the series

$$\sum_{i=1, j=1, k=1}^{\infty, \infty, \infty} a_{ijk}(x, y, z) f_{ijk}(\alpha, \beta, \gamma)$$

will converge uniformly for all positive values of α, β and γ, and for all values of x, y and z in R, and its value, $F(x, y, z, \alpha, \beta, \gamma)$, will approach $f(x, y, z)$ uniformly as α, β and γ approach zero.

3. The summability of the triple Fourier's series at points of continuity of the function developed. Before proving the principal theorem in this connection, it will be necessary to prove several lemmas.

LEMMA 4. *Let R be a region in space, lying within the cube whose sides are $\alpha = \pm(\pi - \varrho_1)$, $\beta = \pm(\pi - \varrho_1)$, $\gamma = \pm(\pi - \varrho_1)$, where ϱ_1 is a small positive quantity, and such that no point of R lies within the sphere whose center is at 0 and whose radius is ϱ_2, where ϱ_2 is also a small positive quantity. Then if $\varphi(\alpha, \beta, \gamma)$ is a function that is finite and integrable* in the region R, the limit*

$$(44) \quad \lim_{l, m, n \to \infty}\left[\frac{1}{lmn}\int\int\int_R \varphi(\alpha, \beta, \gamma)\frac{\sin^2 l\alpha}{\sin^2 \alpha}\frac{\sin^2 m\beta}{\sin^2 \beta}\frac{\sin^2 n\gamma}{\sin^2 \gamma}\, d\alpha\, d\beta\, d\gamma\right]$$

will exist and be equal to zero.

* Here and elsewhere throughout this section, when a function is said to be finite and integrable, it is meant that the function has an integral according to the definition of Lebesgue.

Represent by M the upper limit of the absolute value of R, and by ϱ the smaller of the two quantities ϱ_1 and $\varrho_2/\sqrt{3}$. Then. if ε is an arbitrarily small positive quantity, we can take a positive integer q, such that

$$(45)\qquad \frac{8\pi^3 M}{q\sin^2\varrho} < \frac{\varepsilon}{3}.$$

We shall show that for values of l, m and n greater than q, the expression in brackets in (44) is less than ε, and our lemma will be proved.

Divide the region R into two parts, R_1 and R_1', such that R_1 contains all the points for which $\alpha^2+\beta^2 < 2\varrho_2^2/3$, and R_1' all points for which $\alpha^2+\beta^2 \geqq 2\varrho_2^2/3$. If there are no points for which the first of these two inequalities holds, the region R_1' will coincide with R.

Since no point in the region R lies within the sphere of radius ϱ_2, whose center is at the origin, we have $\alpha^2+\beta^2+\gamma^2 \geqq \varrho_2^2$, and for points in R_1, $\alpha^2+\beta^2 < 2\varrho_2^2/3$; it follows on subtracting the second inequality from the first that $\gamma^2 > \varrho_2^2/3 \geqq \varrho^2$, and hence $|\gamma| \geqq \varrho$ in R_1.

Dividing R_1' into two regions, R_2 and R_3, such that R_2 contains all points for which $|\alpha| < \varrho_2/\sqrt{3}$, and R_3 all points for which $|\alpha| \geqq \varrho_2/\sqrt{3}$, and proceeding as before, we find that, for points in R_2, $|\beta| \geqq \varrho$, and, for points in R_3, $|\alpha| \geqq \varrho$.

We then have

$$\left|\frac{1}{lmn}\iiint\limits_{R_1} \varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right|$$

$$(46)\qquad \leqq \frac{1}{lmn}\iiint\limits_{R_1}\left|\varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right|$$

$$< \frac{M}{lmn\sin^2\varrho}\iiint\limits_{R_1}\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\,d\alpha\,d\beta\,d\gamma.$$

From Fejér's theorem* that

$$\frac{\pi}{2} = \frac{1}{n}\int_0^{\pi/2}\frac{\sin^2 n\alpha}{\sin^2\alpha}\,d\alpha,$$

together with (45) and (46), we have

$$(47)\qquad \left|\frac{1}{lmn}\iiint\limits_{R_1}\varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right| < \frac{8\pi^3 M}{n\sin^2\varrho} < \frac{\varepsilon}{3}, \quad (n \geqq q).$$

* L. c. p. 55.

If there are no points such that $\alpha^2 + \beta^2 < 2\varrho_2^2/3$, the inequality (47) still holds.

By similar methods, we find that analogous equations hold for R_2 $(m \geqq q)$, and for R_3 $(1 \geqq q)$. Combining these three equations, we have

$$\left|\frac{1}{lmn}\int\int\int_R \varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right| < \epsilon, \quad (l, m, n \geqq q),$$

and, as pointed out before, the lemma is proved.

LEMMA 5. *If g, g_1, h, h_1, k, k_1, are positive numbers less than π, the limit*

$$\lim_{l,m,n\to\infty}\left[\frac{1}{lmn\pi^3}\int_{-g_1}^{g}\int_{-h_1}^{h}\int_{-k_1}^{k}\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right] \tag{48}$$

will exist and be equal to unity.

The expression in brackets in (48) may be written in the form

$$\begin{aligned}&\frac{1}{lmn\pi^3}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\\ &\pm\frac{1}{lmn\pi^3}\int\int\int_R\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma,\end{aligned} \tag{49}$$

where R is the region that must be added to, or subtracted from, the cube whose sides are $\alpha = \pm\frac{\pi}{2}$, $\beta = \pm\frac{\pi}{2}$, $\gamma = \pm\frac{\pi}{2}$ to produce the parallelopiped whose sides are $\alpha = g$, $\alpha = -g_1$, $\beta = h$, $\beta = -h_1$, $\gamma = k$, $\gamma = -k_1$. Since this region R satisfies the requirements of the region in Lemma 4, it follows that the second term of (49) approaches zero as l, m and n become infinite. Hence we need only to evaluate the first term, which may be written in the form

$$\left\{\frac{1}{l\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 l\alpha}{\sin^2\alpha}\,d\alpha\right\}\left\{\frac{1}{m\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 m\beta}{\sin^2\beta}\,d\beta\right\}\left\{\frac{1}{n\pi}\int_{-\pi/2}^{\pi/2}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\gamma\right\},$$

which we know from Fejér's work to be equal to unity when l, m and n are positive integers. Hence its limit as l, m and n become infinite is unity, and

since the limit of the second term of (49) is zero, it follows that the limit of the entire expression (49) as l, m and n become infinite is unity.

LEMMA 6. *Let R be a region in space, lying within the cube whose sides are $\alpha = \pm(\pi-\varrho_1)$, $\beta = \mp(\pi-\varrho_1)$, $\gamma = \pm(\pi-\varrho_1)$, where ϱ_1 is a small positive quantity, and such that the point $\alpha = 0$, $\beta = 0$, $\gamma = 0$ lies within or on the boundary of R. Then, if $\varphi(\alpha, \beta, \gamma)$ is a function that is finite and integrable in R, the limit*

$$(50) \qquad \lim_{l,m,n\to\infty}\left[\frac{1}{lmn}\int\int\int_R \varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right]$$

will exist and be equal to zero, provided

$$(51) \qquad \lim_{\alpha,\beta,\gamma\to+0}[\varphi(\alpha,\beta,\gamma)] = 0.$$

In view of (51) we may choose a quantity ϱ_2, so small that

$$(52) \qquad |\varphi(\alpha,\beta,\gamma)| < \frac{\varepsilon}{2\pi^3} \qquad (\alpha^2+\beta^2+\gamma^2<\varrho_2^2),$$

where ε is an arbitrarily small positive quantity. Now divide R into two parts, R_1 and R_2, where R_1 is the sphere with its center at the origin and radius ϱ_2, or as much of it as is included in R, and R_2 is the remainder of R. Then, from Lemma 4, it follows that

$$\lim_{l,m,n\to\infty}\left[\frac{1}{lmn}\int\int\int_{R_2} \varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right] = 0,$$

and hence we can find a q so large that

$$(53) \left|\frac{1}{lmn}\int\int\int_{R_2} \varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right| < \frac{\varepsilon}{2} \quad (l,m,n \geqq q).$$

Using (52) and Fejér's theorem,* we have

$$(54) \left|\frac{1}{lmn}\int\int\int_{R_1} \varphi(\alpha,\beta,\gamma)\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right|$$
$$\leqq \frac{\varepsilon}{2\pi^3}\left\{\frac{1}{lmn}\int\int\int_{R_1}\frac{\sin^2 l\alpha}{\sin^2\alpha}\frac{\sin^2 m\beta}{\sin^2\beta}\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\,d\beta\,d\gamma\right\} < \frac{\varepsilon}{2}.$$

* See previous footnote.

Combining (53) and (54), we find that the absolute value of the quantity in brackets in (50) is less than ϵ, (l, m, $n \geqq q$), and hence the limit (50) exists and is equal to zero.

We wish now to consider the summability of the development of a function of three variables, $f(x, y, z)$, in a triple Fourier's series, i. e., the summability of the triple series

$$(55)\quad \sum_{l=1, m=1, n=1}^{\infty, \infty, \infty} \frac{1}{2^{E(1/l)+E(1/m)+E(1/n)}\pi^3} \cdot \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x', y', z')\, P_{lmn}(x, y, z, x', y', z')\, dx'\, dy'\, dz',$$

where

$$(56)\quad P_{lmn}(x, y, z, x', y', z') = \cos[(l-1)(x'-x)]\cos[(m-1)(y'-y)]\cos[(n-1)z'-z)],$$

$E(s)$ representing the largest integer contained in s.

THEOREM III. *If the function $f(x, y, z)$ is finite and integrable in the region*

$$(57)\quad (-\pi \leqq x \leqq \pi, \quad -\pi \leqq y \leqq \pi, \quad -\pi \leqq z \leqq \pi),$$

the development of the function in a triple Fourier's series will be summable ($C\,1$) to the value of the function at every interior point of the region (57) at which the function is continuous.

For the series (55) we have*

$$\frac{S^{(1)}_{lmn}(x, y, z)}{A^{(1)}_{lmn}} = \frac{S^{(1)}_{lmn}(x, y, z)}{lmn}$$

$$= \frac{1}{8\, lmn\, \pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x', y', z') \left(\frac{\sin\dfrac{l(x'-x)}{2}}{\sin\dfrac{x'-x}{2}}\right)^2 \cdot \left(\frac{\sin\dfrac{m(y'-y)}{2}}{\sin\dfrac{y'-y}{2}}\right)^2 \left(\frac{\sin\dfrac{n(z'-z)}{2}}{\sin\dfrac{z'-z}{2}}\right) dx'\, dy'\, dz'.$$

* The reductions involved in arriving at this value, are exactly analogous to those for the simple series. See Fejér, l. c., p. 54.

Making the transformation

$$(58)\qquad x' - x = 2\alpha, \qquad y' - y = 2\beta, \qquad z' - z = 2\gamma,$$

we have

$$(59)\quad \frac{S^{(1)}_{lmn}(x, y, z)}{lmn} = \frac{1}{lmn\pi^3}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}}\int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}}\int_{-\frac{\pi+z}{2}}^{\frac{\pi-z}{2}} f(x+2\alpha,\ y+2\beta,\ z+2\gamma)$$

$$\cdot\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\, d\beta\, d\gamma.$$

If we can show that the right-hand side of (59) approaches $f(x, y, z)$ as a limit as l, m and n become infinite, at any interior point of the region (57) at which $f(x, y, z)$ is continuous, our theorem will be proved.

Let

$$(60)\qquad \varphi(\alpha, \beta, \gamma) = f(x+2\alpha,\ y+2\beta,\ z+2\gamma) - f(x, y, z).$$

Then the right-hand side of (59) may be written

$$(61)\quad \begin{aligned} &\frac{1}{lmn\pi^3}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}}\int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}}\int_{-\frac{\pi+z}{2}}^{\frac{\pi-z}{2}} \varphi(\alpha, \beta, \gamma)\,\frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\, d\beta\, d\gamma \\ &+\frac{f(x, y, z)}{lmn\pi^3}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}}\int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}}\int_{-\frac{\pi+z}{2}}^{\frac{\pi-z}{2}} \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha\, d\beta\, d\gamma. \end{aligned}$$

From the definition (60), $\varphi(\alpha, \beta, \gamma)$ is finite and integrable in the region of integration of the integrals in (61), and, moreover,

$$\lim_{\alpha, \beta, \gamma \to 0}\,[\varphi(\alpha, \beta, \gamma)] = 0,$$

provided $f(x, y, z)$ is continuous at the point (x, y, z). Hence, from Lemma 6, the first term of (61) approaches zero as l, m and n become infinite, if (x, y, z) is an interior point of the region (57), and if $f(x, y, z)$ is continuous at that point.

From Lemma 5, it follows that the second term of (61) and therefore the entire expression (61), and hence also the right-hand side of (59) approaches $f(x, y, z)$ as a limit as l, m and n become infinite, and therefore the development of the function in the triple Fourier's series is summable $(C1)$ to the value of the function.

COROLLARY. *If $f(x, y, z)$ is finite and integrable in the region* (57), *its Fourier's development will be uniformly summable to $f(x, y, z)$ throughout any region R' whose boundary is interior to the boundary of a region of continuity of $f(x, y, z)$.*

By making slight modifications in Lemmas 4 and 6 and Theorem III, this corollary is easily established.

THEOREM IV. *If $f(x, y, z)$ satisfies the conditions of Theorem III, then for its Fourier's development*

$$L < \frac{S_{lmn}^{(1)}(x, y, z)}{lmn} < M, \qquad \begin{pmatrix} l, m, n = 1, 2, 3, \ldots \\ -\pi \leqq x \leqq \pi, \\ -\pi \leqq y \leqq \pi, \\ -\pi \leqq z \leqq \pi, \end{pmatrix},$$

where L and M are the lower and upper limits respectively of $f(x, y, z)$ in the region (57).

From equation (59) we have

$$\begin{aligned} &\frac{L}{lmn\pi^3} \int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}} \int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}} \int_{-\frac{\pi+z}{2}}^{\frac{\pi-z}{2}} \frac{\sin^2 l\alpha}{\sin^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\alpha \, d\beta \, d\gamma < \frac{S_{lmn}^{(1)}(x, y, z)}{lmn} \\ (62) \qquad & < \frac{M}{lmn\pi^3} \int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}} \int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}} \int_{-\frac{\pi+z}{2}}^{\frac{\pi-z}{2}} \frac{\sin^2 l\alpha}{\sin^2 \alpha} \frac{\sin^2 m\beta}{\sin^2 \beta} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\alpha \, d\beta \, d\gamma, \end{aligned}$$

but the integral in (62) may be written

$$\left(\frac{1}{l\pi} \int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}} \frac{\sin^2 l\alpha}{\sin^2 \alpha} \, d\alpha \right) \left(\frac{1}{m\pi} \int_{-\frac{\pi+y}{2}}^{\frac{\pi-y}{2}} \frac{\sin^2 m\beta}{\sin^2 \beta} \, d\beta \right) \left(\frac{1}{n\pi} \int_{-\frac{\pi+z}{2}}^{\frac{\pi-z}{2}} \frac{\sin^2 n\gamma}{\sin^2 \gamma} \, d\gamma \right),$$

which is equal to unity,* and therefore the theorem follows at once.

* Fejér, l. c., p. 60.

4. **Application.** We shall now apply our results to a problem in the flow of heat. We wish to determine at any instant the temperature of any point of a rectangular parallelopiped whose initial temperature in known, and whose surface is maintained at the temperature zero. Let a be the length, b the width and c the height of the parallelopiped, and $f(x, y, z)$ a function giving the initial temperature of the parallelopiped at any point, when we take the origin at the lower, forward, left-hand corner of the parallelopiped, and let the x, y and z axes fall on the sides whose lengths are a, b and c respectively. The formal method of building up a solution for this type of problem gives us for the temperature of any point at any time*

$$\sum_{l=1,m=1,n=1}^{\infty,\infty,\infty} u_{lmn}(x, y, z, t)$$

(63)

$$= \frac{8}{abc}\sum_{l=1,m=1,n=1}^{\infty,\infty,\infty} a_{lmn}\, e^{-k\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)t} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c},$$

where k is an essentially positive quantity, and where

(64) $$a_{lmn} = \int_0^a\int_0^b\int_0^c f(x', y', z') \sin\frac{l\pi x'}{a} \sin\frac{m\pi y'}{b} \sin\frac{n\pi z'}{c}\, dx'\, dy'\, dz'.$$

In order to show that the expression (63) really furnishes a solution of the problem we must show that:

(1) the expression (63) converges and defines a continuous function of x, y, z and t, say $u(x, y, z, t)$, in the region

(65) $$0 \leqq x \leqq a, \quad 0 \leqq y \leqq b, \quad 0 \leqq z \leqq c, \quad t > 0;$$

(2) the function $u(x, y, z, t)$ satisfies the equation

(66) $$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

* Carslaw, *Introduction to the mathematical theory of the conductivity of heat in solids*, p. 108.

throughout the region (65); and that

$$(3)\qquad \lim_{\substack{t\to+0\\ x\to x_1,\, y\to y_1\, z\to z_1}} u(x, y, z, t) = f(x_1, y_1, z_1),$$

where (x_1, y_1, z_1) is a point within a region throughout which $f(x, y, z)$ is continuous; and that as t approaches zero through positive values, $u(x, y, z, t)$ remains finite as x, y and z approach the coördinates of any point in the region

$$(67)\qquad 0 \leqq x \leqq a, \qquad 0 \leqq y \leqq b, \qquad 0 \leqq z \leqq c.$$

We shall first show that the series (63) is convergent in the region (65). Due to the presence of the convergence factor $e^{-k\pi^2(l^2/a^2+m^2/b^2+n^2/c^2)t}$ in (63), we have for the general term of the series

$$(68)\qquad |u_{lmn}(x, y, z, t)| < \frac{K}{l^2 m^2 n^2}$$

in the region

$$(69)\qquad 0 \leqq x \leqq a, \quad 0 \leqq y \leqq b, \quad 0 \leqq z \leqq c, \quad 0 < t_0 < t,$$

where K is a positive constant and t_0 an arbitrarily small positive constant, provided $f(x, y, z)$ is finite and integrable in (67). The right-hand side of (68) is the general term of a convergent triple series of positive terms, and hence the series (63), of which the left-hand side of (68) is the general term, is absolutely convergent in the region (69), and, since t_0 is arbitrary, in the region (65).

The terms of (63) are functions of x, y and z in the region (69) and hence the series (63) is uniformly convergent throughout the region (69) by virtue of Weierstrass' test, extended for the triple series. Since the terms of the series are continuous throughout the region (69), it follows that the series defines a continuous function throughout that region, and, since t_0 is arbitrary, throughout the region (65). Hence the condition (1) is satisfied.

We shall now show that the condition (2) is satisfied, i.e., that the series (63) satisfies the equation (66). Each term obviously satisfies the equation, and therefore the series will satisfy it provided we have a right to form the derivatives involved in differentiating (63) by differentiating term by term. We may form the derivative of a triple series convergent in a certain region by differentiating term by term if the derived series is uniformly convergent throughout the region and defines a continuous function there. We have shown that the original series converges in the region (65), and it is easily

shown that the derived series is uniformly convergent and continuous in this region by a method analogous to that used in proving the uniform convergence and continuity of the original series in that region. Hence condition (2) is satisfied.

In order to show that condition (3) is satisfied it will be necessary to make use of Theorem II on convergence factors. The convergence factors which occur in the series (63) are functions of one variable only, whereas those involved in Theorem II were functions of three variables, for particular values of l, m and n. We may put the convergence factors of (63) in a form where they are functions of three variables by making the transformations

$$\alpha = \frac{k\pi^2 t}{a^2}, \qquad \beta = \frac{k\pi^2 t}{b^2}, \qquad \gamma = \frac{k\pi^2 t}{c^2}.$$

The convergence factors then have the form

$$e^{-(l^2\alpha+m^2\beta+n^2\gamma)}. \tag{70}$$

We shall first prove the first part of condition (3), i. e., that, as t approaches zero through positive values and as x, y and z approach x_1, y_1, z_1, where (x_1, y_1, z_1) is a point of continuity of the function, the function $u(x, y, z, t)$ approaches $f(x_1, y_1, z_1)$. In order to prove this we shall have to show that the series (63), without the convergence factors, satisfies the conditions of Theorem II, and that the convergence factors (70) satisfy conditions (a)—(d) of Lemma 3 and (e) and (f) of Theorem II. It will then follow from the corollary to Theorem II that $u(x, y, z, t)$ has the desired property.

It follows from the corollary to Theorem III, by a change of variable, that the series (63) without the convergence factors is uniformly summable throughout a region whose boundary is interior to that of a region of continuity of $f(x, y, z)$, and the condition on the series is satisfied.

Hence it remains only to show that the convergence factors (70) satisfy the conditions (a)—(i) inclusive of Theorem II. It is obvious that conditions (d), (e) and (f) are satisfied, and hence only conditions (a), (b), (c), (g), (h) and (i) remain.

Consider first condition (c). Since

$$\left| e^{-(l^2\alpha_0+m^2\beta_0+n^2\gamma_0)} \right| < \frac{K}{l^4 m^4 n^4}, \qquad (l, m, n = 1, 2, 3, \ldots), \tag{71}$$

where α_0, β_0, γ_0 and K are positive constants, we have

$$\sum_{k=1}^{\infty} k \left| e^{-(l^2\alpha_0+m^2\beta_0+k^2\gamma_0)} \right| < \frac{K}{l^4 m^4} \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{K_1}{l^4 m^4},$$

where K_1 is a positive constant. Hence

$$\lim_{l,m\to\infty} l\,m \sum_{k=1}^{\infty} k \left| e^{-(l^2\alpha+m^2\beta+k^2\gamma)} \right| = 0, \qquad (\alpha,\beta,\gamma>0).$$

The proof for condition (b) is carried out in a manner analogous to the above proof for condition (c).

Consider now condition (a). We have

$$\sum_{\lambda_1,\mu_1,\nu_1}^{\lambda_2,\mu_2,\nu_2} i\,j\,k \left| \triangle_{\,2}^{\,2\,2} e^{-(i^2\alpha+j^2\beta+k^2\gamma)} \right| = \left[\sum_{i=\lambda_1}^{\lambda_2} i\,(e^{-i^2\alpha} - 2\,e^{-(i+1)^2\alpha} + e^{-(i+2)^2\alpha}) \right]$$

$$\cdot \left[\sum_{j=\mu_1}^{\mu_2} j\,(e^{-j^2\beta} - 2\,e^{-(j+1)^2\beta} + e^{-(j+2)^2\beta}) \right] \cdot \left[\sum_{k=\nu_1}^{\nu_2} k\,(e^{-k^2\gamma} - 2\,e^{-(k+1)^2\gamma} + e^{-(k+2)^2\gamma}) \right],$$

and therefore condition (a) will hold for some K such that

$$(72) \qquad \sum_{i=1}^{\infty} i\,(e^{-i^2u} - 2\,e^{-(i+1)^2u} + e^{-(i+2)^2u}) < \sqrt[3]{K}, \qquad (u>0).$$

From the law of the mean it follows that

$$(73)\ e^{-i^2u} - 2\,e^{-(i+1)^2u} + e^{-(i+2)^2u} = e^{-(i+\theta)^2u}\{4\,(i+\theta)^2u^2 - 2\,u\},\ (0<\theta<2).$$

From this we see that the terms of the series on the left-hand side of (72) are negative for all positive integral values of i when $i+2>1/\sqrt{2u}$, and positive for all positive integral values of i when $i>1/\sqrt{2u}$. Therefore the series consists of a group of negative terms followed by two terms whose signs may be plus or minus, followed by a group of positive terms. It is readily seen from (73) that each term of the series (72) remains less than some positive constant M, and hence the inequality (72) holds for some K provided any sequence we may choose, consisting entirely of positive or of negative terms of the series in (72), remains less in absolute value than some positive constant.

Consider such a sequence, giving i all integral values from p to q, where p and q are positive integers. The sum of this sequence is

$$(74) \qquad p\,e^{-p^2u} - (p-1)\,e^{-(p+1)^2u} - (q+1)\,e^{-(q+1)^2u} + q\,e^{-(q+2)^2u},$$

which differs from

$$p\,[e^{-p^2u} - e^{-(p+1)^2u}] - (q+1)\,[e^{-(q+1)^2u} - e^{-(q+2)^2u}] \tag{75}$$

by $e^{-(p+1)^2u} - e^{-(q+2)^2u}$, which can never exceed unity. But by the law of the mean,

$$\left| m\,(e^{-m^2u} - e^{-(m+1)^2u}) \right| < 2\,(m+\theta)^2 u\, e^{-(m+\theta)^2u}, \quad (0<\theta<1), \tag{76}$$

the right-hand side of which always remains less than some positive constant for all positive integral values of m and all positive values of u. Therefore the left-hand side of (76), and therefore also (75) and (74), remains less than some positive constant for all values of $u > 0$; hence, as pointed out before, (72) holds, and hence the condition (a) is satisfied.

It is obvious from the fact that expression (74) is finite and from condition (f) that conditions (g) and (h) are satisfied. That condition (i) is satisfied follows from condition (a) and the equation (71). Hence the convergence factors satisfy the conditions of Theorem II, and as pointed out before the first part of condition (c) of this article is satisfied.

If now remains only to prove that, as t approaches zero through positive values and x, y and z approach the coordinates of any point in the region (67), the function $u\,(x, y, z, t)$ remains finite. Since the convergence factors satisfy the conditions of Lemma 3, it follows from that lemma that, for all positive values of α, β and γ or, what is the same thing, for all values of $t > 0$,

$$\sum_{i=1,j=1,k=1}^{\infty,\infty,\infty} S^{(1)}_{ijk}(x, y, z)\, \triangle^{2}_{\frac{2}{2}}\, e^{-(i^2\alpha + j^2\beta + k^2\gamma)} \tag{77}$$

converges to the same value as the series (63), and hence, if we prove that (77) remains fimite for all values of $t > 0$ when x, y and z approach the coördinates of any point in the region (67), we shall have proved that the series (63) satisfies this requirement also.

Since $f(x, y, z)$ is finite and integrable in (67), we have from Theorem IV

$$\left| S^{(1)}_{ijk}(x, y, z) \right| < C\,ijk \qquad \left(\begin{matrix} 0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c \\ i, j, k = 1, 2, 3, \ldots \end{matrix} \right),$$

where C is a positive constant. Making use of this inequality and the fact that the convergence factors (70) satisfy condition (a) of Lemma 3, we have

$$\left|\sum_{i=1,j=1,k=1}^{\infty,\infty,\infty} S_{ijk}^{(1)}(x, y, z) \triangle_{2}^{\frac{2}{2}} e^{-(i^2\alpha+j^2\beta+k^2\gamma)}\right|$$

$$\leqq \sum_{i=1,j=1,k=1}^{\infty,\infty,\infty} \left|S_{ijk}^{(1)}\right| \cdot \left|\triangle_{2}^{\frac{2}{2}} e^{-(i^2\alpha+j^2\beta+k^2\gamma)}\right|$$

$$< \sum_{i=1,j=1,k=1}^{\infty,\infty,\infty} C\, ijk \left|\triangle_{2}^{\frac{2}{2}} e^{-(i^2\alpha+j^2\beta+k^2\gamma)}\right| < CK,$$

$$\left(\begin{matrix} \alpha, \beta, \gamma > 0 \\ 0 \leqq x \leqq a,\ 0 \leqq y \leqq b,\ 0 \leqq z \leqq c \end{matrix}\right),$$

and hence the final condition is satisfied and the series (63) furnishes a solution of the physical problem.

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